

# APPROXIMATING A NORM BY A POLYNOMIAL

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ABSTRACT. We prove that for any norm  $\|\cdot\|$  in the  $d$ -dimensional real vector space  $V$  and for any odd  $n > 0$  there is a non-negative polynomial  $p(x)$ ,  $x \in V$  of degree  $2n$  such that

$$p^{\frac{1}{2n}}(x) \leq \|x\| \leq \binom{n+d-1}{n}^{\frac{1}{2n}} p^{\frac{1}{2n}}(x).$$

Corollaries and polynomial approximations of the Minkowski functional of a convex body are discussed.

## 1. INTRODUCTION AND THE MAIN RESULT

Our main motivation is the following general question. Let us fix a norm  $\|\cdot\|$  in a finite dimensional real vector space  $V$  (or, more generally, the Minkowski functional of a convex body in  $V$ ). Given a point  $x \in V$ , how fast can one compute or approximate  $\|x\|$ ? For example, various optimization problems can be posed this way. As is well known, (see, for example, Lecture 3 of [1]), any norm in  $V$  can be approximated by an  $\ell^2$  norm in  $V$  within a factor of  $\sqrt{\dim V}$ . From the computational complexity point of view, an  $\ell^2$  norm of  $x$  is just the square root of a positive definite quadratic form  $p$  in  $x$  and hence can be computed “quickly”, that is, in time polynomial in  $\dim V$  for any  $x \in V$  given by its coordinates in some basis of  $V$ . Note, that we do not count the time required for “preprocessing” the norm to obtain the quadratic form  $p$ , as we consider the norm fixed and not a part of the input. It turns out that by employing higher degree forms  $p$ , we can improve the approximation: for any  $c > 0$ , given an  $x \in V$ , one can approximate  $\|x\|$  within a factor of  $c\sqrt{\dim V}$  in time polynomial in  $\dim V$ . This, and some other approximation results follow easily from our main theorem.

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**(1.1) Theorem.** *Let  $V$  be a  $d$ -dimensional real vector space and let  $\|\cdot\| : V \longrightarrow \mathbb{R}$  be a norm in  $V$ . For any odd integer  $n > 0$  there exists a homogeneous polynomial  $p : V \longrightarrow \mathbb{R}$  of degree  $2n$  such that  $p(x) \geq 0$  and*

$$p^{\frac{1}{2n}}(x) \leq \|x\| \leq \binom{n+d-1}{n}^{\frac{1}{2n}} p^{\frac{1}{2n}}(x)$$

for all  $x \in V$ .

We prove Theorem 1.1 in Section 2.

Let us fix an  $n$  in Theorem 1.1. Then, as  $d$  grows, the value of  $p^{\frac{1}{2n}}(x)$  approximates  $\|\cdot\|$  within a factor of  $c_n \sqrt{d}$ , where  $c_n \approx (n!)^{-\frac{1}{2n}} \approx \sqrt{e/n}$ . Since for any fixed  $n$ , computation of  $p(x)$  takes a  $d^{O(n)}$  time, for any  $c > 0$  we obtain a polynomial time algorithm to approximate  $\|x\|$  within a factor of  $c\sqrt{d}$  (again, we do not count the time required for preprocessing, that is, to find the polynomial  $p$ ).

If we allow  $n$  to grow linearly with  $d$ , we can get a constant factor approximation. Indeed, if we choose  $n = \gamma d$  for some  $\gamma > 0$ , for large  $d$  we have

$$\binom{n+d-1}{n}^{\frac{1}{2n}} \approx \exp\left\{\frac{1}{2} \ln \frac{\gamma+1}{\gamma} + \frac{1}{2\gamma} \ln(\gamma+1)\right\}.$$

Since for any fixed  $\gamma > 0$ , computation of  $p(x)$  takes  $2^{O(d)}$  time, for any  $c > 1$  we can get an algorithm of  $2^{O(d)}$  complexity approximating the value of  $\|x\|$  within a factor of  $c$ .

One can hope that for special norms  $\|\cdot\|$  (for example, ones with a large symmetry group) one can obtain better approximability/computability results due to special features of the polynomials  $p$  (for example, invariance with respect to the action of a large symmetry group). Indeed, the construction of the proof of Theorem 1.1 (see Section 2) preserves, for example, group invariance.

## 2. PROOF OF THEOREM 1.1

Let  $B$  be the unit ball of  $\|\cdot\|$ , so

$$B = \{x \in V : \|x\| \leq 1\}.$$

Hence  $B$  is a centrally symmetric convex compact set containing the origin in its interior.

Let  $V^*$  be the dual space of all linear functions  $f : V \longrightarrow \mathbb{R}$  and let  $C \subset V^*$  be the polar of  $B$ :

$$C = \left\{f \in V^* : f(x) \leq 1 \text{ for all } x \in B\right\}.$$

Hence  $C$  is a centrally symmetric convex compact set containing the origin in its interior. Using the standard duality argument (see, for example, Section 1.6 of [3]), we can write

$$(2.1) \quad \|x\| = \max_{f \in C} f(x).$$

Let

$$W = V^{\otimes n} = \underbrace{V \otimes \dots \otimes V}_{n \text{ times}} \quad \text{and} \quad W^* = (V^{\otimes n})^* = \underbrace{V^* \otimes \dots \otimes V^*}_{n \text{ times}}$$

be the  $n$ -th tensor powers of  $V$  and  $V^*$  respectively.

For vectors  $x \in V$  and  $f \in V^*$  let

$$x^{\otimes n} = \underbrace{x \otimes \dots \otimes x}_{n \text{ times}} \quad \text{and} \quad f^{\otimes n} = \underbrace{f \otimes \dots \otimes f}_{n \text{ times}}$$

denote the  $n$ -th tensor power  $x^{\otimes n} \in W$  and  $f^{\otimes n} \in W^*$  respectively.

By (2.1), we can write

$$(2.2) \quad \|x\|^n = \max_{f \in C} (f(x))^n = \max_{f \in C} f^{\otimes n}(x^{\otimes n}).$$

Let  $D$  be the convex hull of  $f^{\otimes n}$  for  $f \in C$ :

$$D = \text{conv}\{f^{\otimes n} : f \in C\}.$$

Then  $D$  is a convex compact centrally symmetric (we use that  $n$  is odd) subset of  $W^*$  and from (2.2) we can write

$$(2.3) \quad \|x\|^n = \max_{f \in C} f^{\otimes n}(x^{\otimes n}) = \max_{g \in D} g(x^{\otimes n}).$$

Let us estimate the dimension of  $D$ . There is a natural action of the symmetric group  $S_n$  in  $W^*$  which permutes the factors  $V^*$ , so that

$$\sigma(f_1 \otimes \dots \otimes f_n) = f_{\sigma^{-1}(1)} \otimes \dots \otimes f_{\sigma^{-1}(n)}.$$

Let  $\text{Sym}(W^*) \subset W^*$  be the *symmetric part* of  $W^*$ , that is, the invariant subspace of that action. As is known, the dimension of  $\text{Sym}(W^*)$  is that of the space of homogeneous polynomials of degree  $n$  in  $d$  real variables (see, for example, Lecture 6 of [2]). Next, we observe that  $f^{\otimes n} \in \text{Sym}(W^*)$  for all  $f \in V^*$  and, therefore,

$$(2.4) \quad \dim D \leq \dim \text{Sym}(W^*) = \binom{n+d-1}{n}$$

Let  $E$  be the John's ellipsoid of  $D$  in the affine hull of  $D$ , that is the (unique) ellipsoid of the maximum volume inscribed in  $D$ . As is known, (see, for example, Lecture 3 of [1])

$$E \subset D \subset (\sqrt{\dim D})E.$$

Combining this with (2.3), we write

$$\max_{g \in E} g(x^{\otimes n}) \leq \|x\|^n \leq (\sqrt{\dim D}) \max_{g \in E} g(x^{\otimes n})$$

and, by (2.4),

$$(2.5) \quad \max_{g \in E} g(x^{\otimes n}) \leq \|x\|^n \leq \binom{n+d-1}{n}^{\frac{1}{2}} \max_{g \in E} g(x^{\otimes n}).$$

Let

$$q(x) = \max_{g \in E} g(x^{\otimes n}).$$

We claim that  $p(x) = q^2(x)$  is a polynomial in  $x$  of degree  $2n$ . Indeed, let us choose a basis  $e_1, \dots, e_d$  in  $V$  and the dual basis  $f_1, \dots, f_d$  in  $V^*$ , so that  $f_i(e_j) = \delta_{ij}$ . Then  $W$  acquires the basis

$$e_{i_1 \dots i_n} = e_{i_1} \otimes \dots \otimes e_{i_n} \quad \text{for } 1 \leq i_1, \dots, i_n \leq d$$

and  $W^*$  acquires the basis

$$f_{i_1 \dots i_n} = f_{i_1} \otimes \dots \otimes f_{i_n} \quad \text{for } 1 \leq i_1, \dots, i_n \leq d.$$

Geometrically,  $V$  and  $V^*$  are identified with  $\mathbb{R}^d$  and  $W$  and  $W^*$  are identified with  $\mathbb{R}^{dn}$ . Let  $K \subset W^*$  be the Euclidean unit ball defined by the inequality

$$K = \left\{ h \in W^* : \sum_{1 \leq i_1, \dots, i_n \leq d} h_{i_1 \dots i_n}^2 \leq 1 \right\},$$

where  $h_{i_1 \dots i_n}$  is the corresponding coordinate of  $h$  with respect to the basis  $\{f_{i_1 \dots i_n}\}$ . Since  $E$  is an ellipsoid, there is a linear transformation  $T : W^* \rightarrow W^*$  such that  $T(K) = E$ . Let  $T^* : W \rightarrow W$  be the conjugate linear transformation and let  $y = T^*(x^{\otimes n})$ . Hence the coordinates  $y_{i_1 \dots i_n}$  of  $y$  with respect to the basis  $\{e_{i_1 \dots i_n}\}$  are polynomials in  $x$  of degree  $n$ . Then

$$\begin{aligned} q(x) &= \max_{g \in E} g(x^{\otimes n}) = \max_{h \in K} T(h)(x^{\otimes n}) = \max_{h \in K} h(T^*(x^{\otimes n})) \\ &= \max_{h \in K} h(y) = \sqrt{\sum_{1 \leq i_1, \dots, i_n \leq d} y_{i_1 \dots i_n}^2}. \end{aligned}$$

Hence we conclude that  $p(x) = q^2(x)$  is a homogeneous polynomial in  $x$  of degree  $2n$ , which is non-negative for all  $x \in V$  (moreover,  $p(x)$  is seen to be a sum of squares). From (2.5), we conclude that

$$p^{\frac{1}{2n}}(x) \leq \|x\| \leq \binom{n+d-1}{n}^{\frac{1}{2n}} p^{\frac{1}{2n}}(x),$$

as claimed.

### 3. AN EXTENSION TO MINKOWSKI FUNCTIONALS

There is a version of Theorem 1.1 for Minkowski functionals of not necessarily centrally symmetric convex bodies.

**(3.1) Theorem.** *Let  $V$  be a  $d$ -dimensional real vector space, let  $B \subset V$  be a convex compact set containing the origin in its interior and let  $\|x\| = \inf\{\lambda > 0 : x \in \lambda B\}$  be its Minkowski functional. For any odd integer  $n > 0$  there exist a homogeneous polynomial  $p : V \rightarrow \mathbb{R}$  of degree  $2n$  and a homogeneous polynomial  $r : V \rightarrow \mathbb{R}$  of degree  $n$  such that  $p(x) \geq 0$  and*

$$\left(r(x) + \sqrt{p(x)}\right)^{\frac{1}{n}} \leq \|x\| \leq \left(r(x) + \binom{n+d-1}{n} \sqrt{p(x)}\right)^{\frac{1}{n}}$$

for all  $x \in V$ .

*Proof.* The proof follows the proof of Theorem 1.1 with some modifications. Up to (2.4) no essential changes are needed (note, however, that now we have to use that  $n$  is odd in (2.2)). Then, since the set  $D$  is not necessarily centrally symmetric, we can only find an ellipsoid  $E$  (centered at the origin) of  $W^*$  and a point  $w \in D$ , such that

$$E \subset D - w \subset (\dim D)E,$$

see, for example, Lecture 3 of [1]. Then (2.5) transforms into

$$\max_{g \in E} g(x^{\otimes n}) \leq \|x\|^n - w(x^{\otimes n}) \leq \binom{n+d-1}{n} \max_{g \in E} g(x^{\otimes n}).$$

Denoting

$$p(x) = \left(\max_{g \in E} g(x^{\otimes n})\right)^2 \quad \text{and} \quad r(x) = w(x^{\otimes n})$$

we proceed as in the proof of Theorem 1.1. □

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